**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part II. Optimization methods

## 11. Optimization control problems. Vector case

We considered optimization control problems with unique state function and unique control function. However there exist the optimization problems with many states and controls. We will extend our previous results to this case. The optimization problem for the flight of the missile will be considered as an example.

### 11.1. Problem statement

We extend the previous results to the vector case. Consider control system described by the differential equations

  (11.1)

with the initial conditions

 *х*(0) = *х*0 ,(11.2)

where  is the state vector-function,  is vector control, *f* is the given vector-function, the vector  is the given initial state. For all control *u* we can determine the solution *x* of the problem (11.1), (11.2). The control *u* is chosen from the set

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where  is the given closed subset of the *r*-dimensional Euclid space. Consider the value

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where *g* and *h* are given functions.

**Problem 11.1**. *Find the function u = u*(*t*) *from the set U*, *which minimize the value I*.

We obtain Problem 10.2 for   .

### 11.2. Maximum principle

Determine Lagrange function



where  are Lagrange multipliers. If the function *x* is the solution of the equation (11.3), then the value *L* is equal to *I.* Determine the function

 **  (10.3)

Then we have the equality

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Suppose *u* is a solution of our problem (the optimal control). So we have then inequality

 Δ*I* = *I*(*v*,*y*) – *I*(*u*,*x*) ≥ 0 ∀*v*∈*U*, (11.4)

where *х* and *у* are the solutions of the problem (11.1), (11.2) for the controls *u* and *v.* Then previous inequality can be transformed to

 Δ*L* = *L*(*v*,*y*,*р*) – *L*(*u*,*x*,*р*) ≥ 0 ∀*v*∈*U*, ∀*р*  (11.5)

Find the value



where

Δ*H* = *H*(*t*,*v*,*y*,*р*) – *H*(*t*,*u*,*x*,*р*), Δ*h* = *h*[*y*(*T*)] – *h*[*x*(*T*)].

Suppose the functions of the problem statement are smooth enough. Using Taylor formula we get



where , and *η*1 is the high term with respect Δ*х*(*T*). Then we have

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where  *η*2 is the high term with respect Δ*х*,

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So the inequality (13.5) can be transformed to

  (11.6)

where

Δ*uH* = *H*(*t*,*v*,*x*,*р*) – *H*(*t*,*u*,*x*,*р*),



Using the integration by parts we get



because of the equalities *х*(0) = *у*(0) = *х*0. So we obtain the inequality

  (11.7)

Using the arbitrariness of the vector-function *p* chose it such that this inequality transforms to the easy one. Suppose *p* is the solution of the equation

  (11.8)

with final condition

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where  The system (13.8), (13.9) is called the *adjoint system*. It contains *n* differential equations as the state system (11.1). Then the inequality (13.7) is transformed to

  (11.10)

The left side of this inequality is the sum of two values. If the control *v* is close enough to the optimal control *v*, then the high term *η* has the high order of the infinitesimality. Then the sign of the sign is determine by the first term. So we obtain the inequality



We can prove that the sign of the integral is determine by the sign of the value under the integral. So we get the inequality



It can be transform to

  (11.11)

This equality is called the *maximum principle*.

**Theorem 11.1.** *The solution of the Problem* 11.1 *satisfies the maximum principle.*

Hence we have the system (11.1), (11.2), (11.8), (11.9), (11.12) for finding unknown vector-functions *u*, *х*, *р.* The maximum principle is conditional problem of the maximization of the known function *H* with respect to *r* variables (controls). The obtained system can be solved approximately by iterative method as the corresponding optimality condition for Problem 10.2.

### 11.3. Example

Consider the system

  (10.13)

with the initial conditions

  (10.14)

The set of the admissible controls is determine by the inequalities



We have the problem of minimization the value



Determine the functions





Find the derivatives

 



The adjoint system (11.8), (11.9) is transformed to

  (11.15)

  (11.16)

We have the maximum principle



Determine the derivative of the function *H* with respect to the control. It is equal to zero. So we find



Then we find the solution of the maximum principle

  (11.17)

The system of the optimality conditions (11.13) – (11.17) can be solved with using of the iterative method.

### 11.4. Maximization of the length of the missile flight

We return to Problem 10.1. This is the problem of the maximization of the length of the missile flight. We have the system, described by the differential equations

 , (11.18)

with the initial conditions

 . (11.19)

The state functions *x* and *y* are coordinates of the missile here, the constant *a* is the acceleration of the driving force (the parameter of the system), *g* is the gravitational acceleration, and the angle *u = u*(*t*) is the control. We would like to choose the control such that the length of the flight

 *L* = *x*(*T*) + **** (11.20)

will be maximal.

Transform this problem to Problem 11.1. Denote

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Determine the functions

*f*1 = *х*3, *f*2 = *x*4, *f*3 = *a* cos *u*, *f*4 = *a* sin *u* – *g*.

Then the system (11.18) is transformed to the state equations (11.1). The initial conditions (11.19) are transformed to (11.2) with initial state *x*0 = 0. In our problem we have ** For the transformation the maximizing value to the standard form we determine the functions

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Determine the function



Then we determine the adjoint equations

 . (11.21)

Add the corresponding final conditions

 . (11.22)

Now we have the problem of maximization. So the optimal control is the point of the minimum of the function *H*. Determine the function

*F*(*u*) = *a* (*p*3 cos *u* + *p*4 sin *u*).

It includes only the terms of the function *H*, which depends from the control. So we get the problem of the minimization of the function *F* with respect to the control *u*. Using the stationary condition we obtain

*F'*(*u*) = *a* (*p*4 cos *u* – *p*3 sin *u* ) = 0.

This is the algebraic equation with respect to *u*. So we find

  (11.23)

We can find the control if we will know the functions *p*3 and *p*4. It depends from the functions *p*1 and *p*2 because of the first and second equalities (11.21). By third and fourth equalities (11.21) the functions *p*1 and *p*2 are constants. Using final conditions (11.22) we find



Solve the problem (11.21), (11.22); we get



Then we find



Find now the value

  (11.24)

Using formulas (11.23), (11.24) we conclude that the optimal control *u* is constant. It depends from the final values of the state functions *x*2, *x*3, *x*4. We solve the problem (11.18), (11.19) with constant control. We find



Using the equalities (11.23), (11.24), we find



It can be transform to the equality

*g* sin3*u* – 2*a* sin2*u* + *a* = 0.

Denote

*v* = sin*u*, *b* = *a*/*g*.

We obtain the cubic equation

 *ϕ*(*v*) = *v*3 – 2*bv*2 + *b* = 0. (11.25)

Hence the optimal control is determine only by the driving acceleration.

Determine the properties of the function *ϕ*. Find its derivative

*ϕ*'(*v*) = 3*v*2 – 4*bv*.

So this function has two local extremum. It has the local maximum at the point *v* = 0 and the local minimum at the point *v* = 4/3 *b* (see Figure 11.1). Find the values





Figure 11.1. The function *ϕ* = *ϕ*(*b*).

So the equation (11.25) has three solutions. One of them is negative, and third is greater than 4/3*b* (see Figure 11.1). Note that in our problem the acceleration *a* is greater than *g* because the missile need to flight. So the third solution is greater than 1. However it is impossible the parameter *v* is the sinus. If is negative we do not have the flight to the increase of the coordinate *x.* Then the value *v* is the point from unit interval. Find

*ϕ*(0) = *b* > 0 , *ϕ*(1) = 1 – *b* < 0.

Therefore the equation (11.25) has the unique solution *v*\*=*v*\*(*а*) from unit interval. So the solution of our problem is

*u*(*t*) = arcsin *v*\*, *t* ∈(0,*Т*).

**Outcome**

* Necessary conditions of the optimality for the vector optimization control problem consist of the state system, the adjoint system, and Pontryagin’s maximum principle.
* The adjoint system consists of the adjoint systems of differentiation equations and the final conditions; the quantity of these equations is equal to the quantity of the state functions.
* Maximum principle is a problem of the maximization of some function *H* of many variables; its quantity is equal to the quantity if the control functions.
* Necessary conditions of the optimality for the optimization control problem can be solved by iterative method.
* The problem of the length maximization for the flight of the missile can be considered as an application of this theory.

### Task 11. Optimization control problem for a system of differential equations

Consider the control system described by the differential equations



with initial conditions



The set of the admissible control is described by the inequalities



We have the problem of the minimization of the value

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Table of the parameters.

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Steps of the task.

1. Write the concrete problem statement.
2. Determine the function *Н.*
3. Determine the adjoint system.
4. Determine the maximum principle.
5. Find the control from the maximum principle.
6. Write the iterative method for solving the conditions of the optimality.

### Next step

We considered the optimization control problems with free final states. However, there exist practical optimization control problems with fixed final state. We will try to extend these results to this case.